The Four Fundamental Subspaces

Let A be an $m \times n$ matrix.

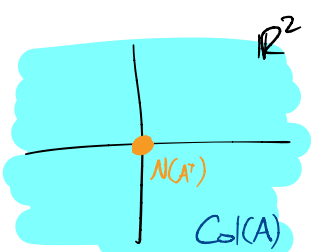
- the nullspace of A is $N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\} \subseteq \mathbb{R}^n$
- the column space of A is $C(A) =$ Span of Columns of $A \subseteq \mathbb{R}^m$
- the row space of A is $C(A^T) =$ Span of Cols of $A^T \subseteq \mathbb{R}^n$
  $=$ Span of Rows of A
- the left nullspace of A is $N(A^T) = \{x \in \mathbb{R}^m \mid A^T x = 0\} \subseteq \mathbb{R}^m$
  $= \{x \in \mathbb{R}^m \mid x^T A = 0\}$

Note: $N(A), C(A^T) \subseteq \mathbb{R}^n$ and $C(A), N(A^T) \subseteq \mathbb{R}^m$.

Ex: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. (So $A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$)

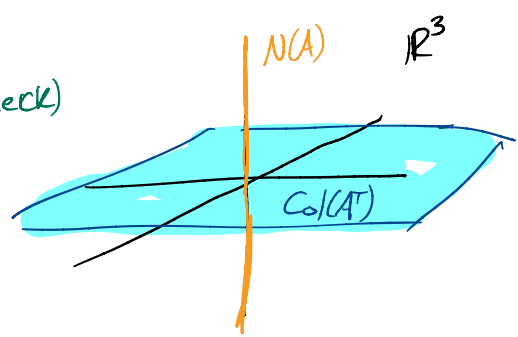
Then $C(A) = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = \mathbb{R}^2$

$N(A^T) = \{0\}$ since $A^T x = 0$ has no free variables



and $N(A) = \text{Span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ (Check)

$C(A^T) = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$



We know: $\dim C(A) = \text{rank}(A)$

$\dim N(A) = n - \text{rank}(A)$

So: $\dim C(A^T) = \text{rank}(A^T)$

$\dim N(A^T) = m - \text{rank}(A^T)$

How are $\text{rank}(A)$ and $\text{rank}(A^T)$ related?

Let's find a basis for the row space.

One basis is the pivot columns of A^T since this is a basis for $C(A^T)$. But we'd like to instead find a basis based on the RREF of A to save time

Key Point 1: Row operations do not change the row space

So: row space of A = row space of RREF, R

Ex: $A = \begin{bmatrix} -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} = R$

Row operations:

(R1 of R) is $-1 \cdot$(R1 of A)

(R2 of R) is $1 \cdot$(R1 of A) $+ 2 \cdot$(R2 of A)

} L.Cs of (R1 of A) and (R2 of A)

Since the rows of R are L.Cs of the rows of A (and vice versa) the row space of R = row space of A.

Key Point 2: The pivot rows of R are a basis for the row space of R

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = R$

$R^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix}$

The first two rows of R (or columns of R^T) are L.I. and span $C(R^T)$

Now since the pivot rows of R form a basis for $C(R^T)$ and $C(R^T) = C(A^T)$, we have that

Basis for $C(A^T)$: the pivot rows of R form a basis for $C(A^T)$

So, $\text{rank}(A^T) = \dim C(A^T) = \#$ pivot rows of A
$= \#$ pivot columns of A
$= \text{rank}(A)$!

Therefore, if A is $m \times n$ and $\text{rank}(A) = r$, then:

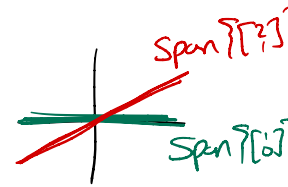
- $\dim C(A) = \dim C(A^T) = r$,
- $\dim N(A) = n - r$
- $\dim N(A^T) = m - r$.

Warning: The pivot rows of R form a basis for $C(A^T)$ but the pivot columns of R do not form a basis for $C(A)$

This is because row operations change the column space (but not the row space)

Eg. $A = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} = R$ (RREF)

$\text{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\} \neq \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$



Geometrically: We can say more about the relationship between $C(A), N(A^T) \subseteq \mathbb{R}^m$ and $C(A^T), N(A) \subseteq \mathbb{R}^n$.

But first, we need to understand the geometry of dot products (coming soon!)

The Big Recap

Let A be $m \times n$.

A) The following statements are equivalent.
(ie. if one is true, then they all are true.
if one is false, then they all are false)

- A has a pivot in every row
- $\text{rank}\, A = m$
- $Ax = b$ has a solution $\forall\, b \in \mathbb{R}^m$ (1 or ∞ sol'ns)
- $C(A) = \mathbb{R}^m$
- The columns of A span $\mathbb{R}^m$
- $T: \mathbb{R}^n \to \mathbb{R}^m$ given by $T(x) = Ax$ is onto (ie. $\text{im}\, T = \mathbb{R}^m$)

B) The following statements are equivalent

- A has a pivot in every column
- $\text{rank}\, A = n$
- $Ax = b$ has 0 or 1 solution $\forall\, b \in \mathbb{R}^m$ (no free variables)
- $Ax = 0$ only has the zero sol'n.
- $N(A) = \{\vec{0}\}$
- The columns of A are linearly independent
- $T: \mathbb{R}^n \to \mathbb{R}^m$ given by $T(x) = Ax$ is one-to-one (ie. $\ker T = \{0\}$)

c) If $m=n$ (A is square), then the following statements are equivalent:

- A has n pivots (a pivot in every row and column.)
- $\text{rank } A = n$
- A is invertible
- RREF of A is I
- $Ax=b$ has a unique soln $\forall\, b \in \mathbb{R}^n$, namely $x=A^{-1}b$
- $Ax=0$ has only the zero solution
- $N(A) = \{0\}$
- $C(A) = \mathbb{R}^n$
- $N(A^T) = \{0\}$
- $C(A^T) = \mathbb{R}^n$
- The columns of A are linearly independent.
- The columns of A span $\mathbb{R}^n$
- The columns of A form a basis for $\mathbb{R}^n$
- $T: \mathbb{R}^n \to \mathbb{R}^n$ given by $T(x)=Ax$ is onto
- $T: \mathbb{R}^n \to \mathbb{R}^n$ given by $T(x)=Ax$ is one-to-one